SUMS of R.V's

- For any r.v's $X_1, X_2, \ldots X_n$ consider

  r.v $W = X_1 + X_2 + \ldots + X_n = \sum_{i=1}^{n} X_i$

- $$E[W] = \sum_{i=1}^{n} E[X_i]$$

  $$Var[W] = \sum_{i=1}^{n} Var[X_i] + 2 \sum_{\substack{i,j=1 \\ i \neq j}}^{n} Cov[X_i, X_j]$$

PDF of SUM of 2 r.v's

- 2 r.v's X, Y with JPDF $f_{X,Y}(x,y)$.

  The pdf of W = X + Y

  $$f_W(w) = \int_{-\infty}^{\infty} f_{X,Y}(x, w-x)\,dx$$

  » Convolution integral

- If X and Y are INDEP. r.v's,

  $$f_W(w) = \int_{-\infty}^{\infty} f_X(x) \,.\, f_Y(w-x)\,dx$$

MOMENT GENERATING FUNCTION (MGF)

- convolution integrals are tough, so it is easier to use transform methods to replace convolution of pdf's --> multiplication of transforms

- MGF of r.v X is $\phi_X(s) = E[e^{sX}]$

- Let Y = a X + b . The MGF of Y :

  $$\phi_Y(s) = \exp(s\,b)\,\phi_X(a\,s)$$

MGF properties

- Let $\phi_X(s)$ be the MGF for r.v X
  The nth moment, $E[X^n]$, is easily determined from the MGF :

$$E\left[X^n\right] = \left[\frac{d^n \phi_X(s)}{ds^n}\right]_{s=0}$$

- For $W = X_1 + X_2 + \ldots + X_n$, the MGF of W

$$\phi_W(s) = \phi_{X_1}(s).\phi_{X_2}(s).\ldots.\phi_{X_n}(s)$$

Sums of INDEP. GAUSSIAN r.v's

- Single Gaussian r.v with mean μ and variance σ^2 . Its MGF is

$$\phi_X(s) = \exp\left[s\mu + \frac{\sigma^2 x^2}{2}\right]$$

- Consider n indep. Gaussian r.v's X_i with means μ_i , variances σ^2_i (i=1…n) , then $W = X_1 + X_2 + \ldots + X_n$ is a Gaussian r.v with mean and variance :

$$E[W] = \sum_{i=1}^{n} E\left[X_i\right] = \sum_{i=1}^{n} \mu_i \qquad Var[W] = \sum_{i=1}^{n} Var\left[X_i\right] = \sum_{i=1}^{n} \sigma_i^2$$

General Comments on sum of iid's

- *iid* = Independent, Identically Distributed r.v's
- Poisson iid's X_i ---> Poisson $W = \sum_{i=1}^{n} X_i$
- Gaussian iid's ---> Gaussian $W = \sum X_i$
- ALL other pdf's will yield a **DIFFERENT** class of pdf for $W = \sum X_i$

Random Sums of iid's

- Sequence of *N* iid's $X_1, X_2, \ldots X_N$, where *N* is **also** a *random* number
- The MGF for $R = X_1 + X_2 + \ldots + X_N$ is

$$\phi_R(s) = \phi_N(\ell n\, \phi_X(s))$$

since each X_i has the same MGF $\phi_X(s)$.

- $$E[R] = E[N].E[X]$$

$$Var[R] = E[N].Var[X] + Var[N].(E[X])^2$$

Central Limit Theorem

- **Theorem** : $\{X_i\}$ sequence of iid r.v with mean μ_X and variance σ_X^2. Consider the normalized r.v

$$Z_n = \frac{\sum_{i=1}^{n} X_i - n\,\mu_X}{\sqrt{n}\,.\,\sigma_X}$$

i.e

$$E[Z_n] = 0\,, \quad Var[Z_n] = 1$$

then *the CF*

$$\lim_{n\to\infty} F_{Z_n}(z) = \Phi(z)$$

i.e., approaches a Gaussian CDF

Central Limit Theorem Approximation

- $\{X_i\}$ sequence of iid r.v with mean μ_X and variance σ_X^2.
  Let $W = X_1 + X_2 + \ldots + X_n$
  The CDF of W *(irrespective of the PDF of the X_i)*

$$F_W(w) = \Phi\left(\frac{w - n\,\mu_X}{\sqrt{n}\,.\,\sigma_X}\right)$$